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UPPER AND LOWER BOUNDS TO THE NATURAL FREQUENCIES OF VIBRATION OF CLAMPED RECTANGULAR ORTHOTROPIC PLATES

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Abstract—The Rayleigh–Ritz technique, using clamped beam eigen functions, has been employed to determine the upper bounds for the eigen values for a clamped orhtotropic plate. The decomposition technique after Bazely and Fox has been used to estimate the lower bounds for the first few natural frequencies. The estimates for the upper bounds have been evaluated for all modes by not imposing any restriction on the symmetry conditions. Variations of the first two natural frequencies for various rigidity and aspect ratios which can be of some use to the designers are presented. Also the upper and lower bounds for the first few natural frequencies are tabulated. Comparison of the results for special cases with other reported data have been made whenever such results are available.

1. INTRODUCTION

Unlike rectangular plates with two opposite edges or all the four edges simply supported, a plate with clamped boundaries does not have an "exact" or closed form solution. Thus, a considerable amount of work has been done to evaluate the natural frequency(ies) using approximate techniques. The Rayleigh–Ritz energy method has proven to be the most popular technique with considerable success demonstrated by numerous works.

Hearmon[1] investigated the frequency of vibration of rectangular wood and plywood plates. He obtained approximations for the fundamental natural frequency of specially orthotropic, rectangular plates with clamped edges using the Rayleigh method. His assumed deflection expression consistent with the clamped boundaries, consisted of a product of similar fourth order polynomials in the x and y directions. He further modified his results using a Ritz modification with a two term deflection function. Reddy and Rajappa[2] obtained the same fundamental frequency expression by solving certain interconnected beam systems with elastic equivalence to orthoropic plates and various boundary conditions. They used Galerkin's formulation of the variational principle. Lekhnitski[3] assumed products of trigonometric functions for the deflection and used the Rayleigh method to evaluate the natural frequency of orthotropic plates. His results are somewhat higher than Hearmon's results. Iyengar and Jagadish[4] obtained numerical results for the free vibration of clamped orthotropic plates and reported more accurate frequencies for various rigidities than Kanazawa and Kawai[5].

Huffington[6] and Young[7] considered the modal patterns in rectangular orthotropic plates and reported the existence of non-parallel nodal lines for clamped orthotropic plates.

Dickinson[8] considered the vibration of a rectangular plate with various boundary conditions. He extended the sine series solution for isotropic plates developed by Dill and Pister[9] to an orthotropic case. Dickinson's first three natural frequencies are lower than Hearmon's upper bounds.

Bert and Mayberry[10] employed the Rayleigh–Ritz technique and used beam functions to determine the natural frequencies of clamped laminated anisotropic plates. They have compared the theoretical results with that of experimental ones. Lin and King[11] studied the free vibration of unsymmetric cross ply and antisymmetric angle-ply plates. They have used an approximate technique due to Bolotin to obtain the natural frequencies.

Bazely *et al.*[12], computed the lower bounds of the first fifteen frequencies of a rectangular isotropic plate with clamped edges by using a decomposition technique.

This technique developed by Bazely and Fox[13] decomposes the governing differential equation to two or more equations that are individually resolvable. However, Bazely, Fox and Stadtler's treatment is limited to a particular symmetry class.

In the present work, an attempt has been made to extend the Bazely et al. method to orthotropic plates with clamped edges but without symmetry limitations. In addition, as opposed to other solution methods providing only approximate results, both upper and lower bound estimates are presented in order to accurately bracket the true eigenvalue for the first few natural frequencies of each case considered. This bracketing technique permits the results to be accepted with a much higher degree of confidence than other single valued approximation solution methods.

2. THEORY

2.1 *Equation of motion*

For a freely vibrating clamped "specially orthotropic" thin plate, (Fig. 1) the equation of motion is given by[14]

$$D_x\frac{\partial^4 w}{\partial x^4}+D_y\frac{\partial^4 w}{\partial y^4}+H\frac{\partial^4 w}{\partial x^2\partial y^2}+\rho h\frac{\partial^2 w}{\partial t^2}=0 \tag{2.1}$$

with the boundary conditions

$$w=\frac{\partial w}{\partial x}=0 \quad \text{at} \quad x=0,\ a$$

$$w=\frac{\partial w}{\partial y}=0 \quad \text{at} \quad y=0,\ b \tag{2.2}$$

where

$$H=D_x\nu_{yx}+D_y\nu_{xy}+4D_{xy}$$
$$D_x=E_xh^3/12\mu$$
$$D_y=E_yh^3/12\mu$$
$$D_{xy}=G_{xy}h^3/12$$
$$\mu=1-\nu_{xy}\nu_{yx}$$

and ρ, ν, D being the mass density, Poisson's ratio, and rigidity respectively.

The solution to the equation of motion can be written as

$$w(x,y,t)=\psi(x,y)\cos(\omega t+\phi). \tag{2.3}$$

Substituting this in equation (2.1), the equation of motion reduces to

$$D_x\frac{\partial^4\psi}{\partial x^4}+D_y\frac{\partial^4\psi}{\partial y^4}+H\frac{\partial^4\psi}{\partial x^2\partial y^2}-\rho h\omega^2\psi=0. \tag{2.4}$$

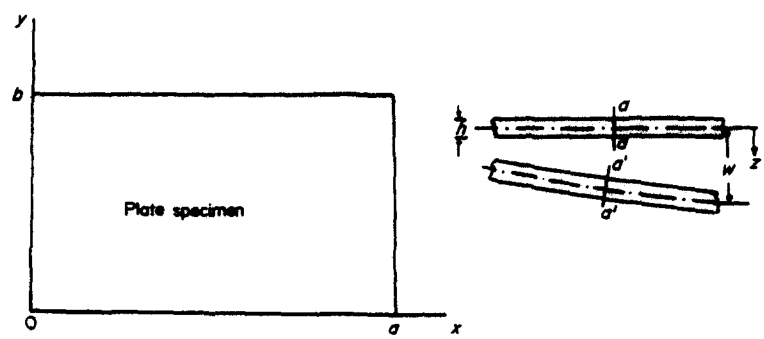

Fig. 1. Coordinate system used in analysis.

Letting $X = x/a$ and $Y = y/b$, eqns (2.4) becomes

$$\frac{D_x}{a^4}\frac{\partial^4\psi}{\partial X^4}+\frac{D_y}{b^4}\frac{\partial^4\psi}{\partial Y^4}+\frac{H}{a^2b^2}\frac{\partial^4\psi}{\partial X^2\partial Y^2}-\rho h\omega^2\psi=0 \tag{2.5}$$

and eqn (2.2) can be written as

$$\begin{aligned}
&\psi(0,Y)=0, && \psi(1,Y)=0\\
&\psi(X,0)=0, && \psi(X,1)=0\\
&\frac{\partial\psi}{\partial X}(0,Y)=0, && \frac{\partial\psi}{\partial X}(1,Y)=0\\
&\frac{\partial\psi}{\partial Y}(X,0)=0, && \frac{\partial\psi}{\partial Y}(X,1)=0.
\end{aligned} \tag{2.6}$$

Now defining $\lambda = \rho h\omega^2 a^4/D_x$, the governing equation becomes

$$\frac{\partial^4\psi}{\partial X^4}+\frac{H}{D_x}\left(\frac{a}{b}\right)^2\frac{\partial^4\psi}{\partial X^2\partial Y^2}+\frac{D_y}{D_x}\left(\frac{a}{b}\right)^4\frac{\partial^4\psi}{\partial Y^4}-\lambda\psi=0. \tag{2.7}$$

Equation (2.7) along with eqn (2.6) constitute the eigenvalue problem for an orthotropic plate with clamped edges.

2.2 *Rayleigh–Ritz method for upper bounds*

Let

$$L=\frac{\partial^4}{\partial X^4}+P\frac{\partial^4}{\partial X^2\partial Y^2}+Q\frac{\partial^4}{\partial Y^4} \tag{2.8}$$

where

$$P=\frac{Ha^2}{D_xb^2},\quad Q=\frac{D_ya^4}{D_xb^4}.$$

The above operator L can be easily shown to be self adjoint and positive definite. The concept of this method consists of determining the stationary values of the Rayleigh quotient

$$R(\psi)=\frac{(\psi,L\psi)}{(\psi,\psi)}=\bar{\lambda} \tag{2.9}$$

where the parenthesis refer to the inner product of the arguments. The inner product is defined here as

$$(u,v)=\int_0^1\int_0^1 uv\,\mathrm{d}X\,\mathrm{d}Y.$$

The solution is not for all admissible functions ψ, but for the linear manifold spanned by an arbitrary, finite set of linearly independent functions $[\psi]$ satisfying the geometric boundary conditions of the operator L[15]. The problem then consists of finding the functions ψ of the form

$$\psi=\sum_{i=1}^{n}a_i\psi_i. \tag{2.10}$$

Substituting this in eqn (2.9), the equation in matrix form becomes

$$[(\psi_i, L\psi_j)][a_j]=\bar{\lambda}[(\psi_i,\psi_j)][a_j]. \tag{2.11}$$

By choosing a set of orthonormal functions, $\{\psi_i\}$ eqn (2.11) reduces to

$$[(\psi_i, L\psi_j)][a_j] = \bar{\lambda}[I][a_j]. \tag{2.12}$$

The function ψ can be expressed in terms of two separate beam functions as

$$\psi = A_{ik}\phi_i(X)\phi_k(Y). \tag{2.13}$$

The A_{ik} term is a numerical coefficient whereas ϕ is the normalized eigenfunction of an unloaded isotropic clamped beam and is given by[16]

$$\phi_m = \cosh \epsilon_m z - \cos \epsilon_m z - E_m(\sinh \epsilon_m z - \sin \epsilon_m z) \tag{2.14}$$

where E_m is the normalizing function given by

$$E_m = \frac{\cosh \epsilon_m - \cos \epsilon_m}{\sinh \epsilon_m - \sin \epsilon_m}$$

and ϵ_m are the eigenvalues of the clamped beam differential equation

$$\frac{d^4 \phi_m}{d z^4} - \epsilon_m{}^4 \phi_m = 0 \tag{2.15}$$

determined by the roots of the characteristic equation

$$\cosh \epsilon_l \cos \epsilon_l = 1. \tag{2.16}$$

To enhance the handling of the required inner product, the eqn (2.13) will be redefined as

$$\psi_i^k = A_i^k \phi_i(X)\phi^k(Y). \tag{2.17}$$

Now

$$\begin{aligned}(\psi_i^k, L\psi_j^l) &= \int_0^1 \int_0^1 \left[\psi_i^k \frac{\partial^4 \psi_j^l}{\partial X^4} + \psi_i^k P \frac{\partial^4 \psi_j^l}{\partial X^2 \partial Y^2} + \psi_i^k Q \frac{\partial^4 \psi_j^l}{\partial Y^4}\right] dX\, dY \\ &= \epsilon_i^4 + P \int_0^1 \int_0^1 \psi_i^k \frac{\partial^4 \psi_j^l}{\partial X^2 \partial Y^2} dX\, dY + Q\epsilon_l^4 \quad \text{for} \quad i = j,\ k = l \\ &= P \int_0^1 \int_0^1 \psi_i^k \frac{\partial^4 \psi_j^l}{\partial X^2 \partial Y^2} dX\, dY \quad \text{for} \quad i \neq j,\ k \neq l. \end{aligned} \tag{2.18}$$

Using the boundary conditions and integrating by parts, eqn (2.18) becomes

$$(\psi_i^k, L\psi_j^l) = \epsilon_i^4 + \frac{P}{\epsilon_i^4 \epsilon_k^4}[R(i) - S(i, i)] + [R(k) - S(k, k)] + Q\epsilon_k^4 \quad \text{for} \quad i = j,\ k = l \tag{2.19a}$$

$$= \frac{P}{\epsilon_i^4(\epsilon_k^4 - \epsilon_l^4)}[R(i) - S(i, i)][S(k, l) - S(l, k)] \quad \text{for} \quad i = j,\ k \neq l \tag{2.19b}$$

$$= \frac{P}{\epsilon_k^4(\epsilon_i^4 - \epsilon_j^4)}[S(i, j) - S(j, l)][R(k) - S(k, k)] \quad \text{for} \quad i \neq j,\ k = l \tag{2.19c}$$

$$= \frac{P}{(\epsilon_i^4 - \epsilon_j^4)(\epsilon_k^4 - \epsilon_l^4)}[S(i, j) - S(j, i)][S(k, l) - S(l, k)] \quad \text{for} \quad i \neq j,\ k \neq l \tag{2.19d}$$

where

$$R(m) = \left[\left(\frac{d^3 \phi_m}{d z^3}\right)^2\right]_{z=1}$$

and

$$S(m, n) = \left[\frac{d^2 \phi_m}{d z^2} \frac{d^3 \phi_n}{d z^3}\right]_0^1.$$

Equations (2.19) represent the coefficient matrix for the standard eigenvalue problem formulation defined by eqn (2.12). The solution of this eigenvalue problem provides the upper bounds for the natural frequencies of a clamped orthotropic plate.

2.3 *Decomposition method for lower bounds*

This method requires the operator L to be decomposable into a sum of quadratic forms [13].

$$L = \sum_{\alpha=1}^{m} A_\alpha \tag{2.20}$$

such that each eigenvalue problem

$$A_\alpha \psi - \lambda B \psi = 0 \tag{2.21}$$

is explicitly resolvable. Each set of eigenvalues for a given variation problem α, designated as λ_i^α, are taken to be enumerated in increasing order, as

$$\lambda_1^\alpha \leq \lambda_2^\alpha \leq \lambda_3^\alpha \ldots\ldots, \quad \alpha = 1, 2, 3, \ldots . M. \tag{2.22}$$

The lower bounds of the true eigenvalues are designated by λ_μ^n and depend on the M-tuple n of positive integern n_α,

$$n = (n_1, n_2, \ldots, n_M).$$

The lower bounds are obtained in order from solutions of the symmetric eigenvalue problem,

$$[\lambda^n][I] - [D] \tag{2.23}$$

where

$$\lambda^n = \sum_{\alpha=1}^{M} \lambda_{n_\alpha+1}^\alpha$$

and $[D]$ of the order $|n|(=\sum_1^M n_\alpha)$ has the form

$$D = \begin{bmatrix} D^{11} D^{12} \ldots . D^{1M} \\ D^{21} D^{22} \quad . \\ . \quad . \\ . \quad . \\ D^{M1} \ldots\ldots\ldots D^{MM} \end{bmatrix}.$$

The submatrices of D are obtained from the expression

$$D^{\alpha\beta} = \{(\lambda_{n_\alpha+1}^\alpha - \lambda_\mu^\alpha)^{1/2}(\psi_\mu^\alpha, \psi_\nu^\beta)(\lambda_{n_\beta+1}^\beta - \lambda_\nu^\beta)^{1/2}\}. \tag{2.24}$$

The lower bounds calculated by this method increases monotonically with each index n_α and satisfy the inequalities

$$\sum_{\alpha=1}^{M} \lambda_1^\alpha \leq \lambda_\mu^n \leq \lambda^n, \quad \mu = 1, 2, \ldots . |n|$$

and give bounds according to

$$\lambda_\mu^n \leq \lambda_\mu, \mu = 1, 2, \ldots\ldots, |n|. \tag{2.25}$$

Equation (2.7) can be decomposed into two parts as

$$\frac{\partial^4\psi_1}{\partial X^4}+\frac{D_y}{D_x}\frac{a^4}{b^4}\frac{\partial^4\psi_1}{\partial Y^4}-\lambda_1\psi_1=0 \tag{2.26}$$

and

$$\frac{H}{D_x}\frac{a^2}{b^2}\frac{\partial^4\psi_2}{\partial X^2\partial Y^2}-\lambda_2\psi_2=0. \tag{2.27}$$

The boundary conditions for the above two equations are obtained from eqn (2.6).

Using separation of variables, and eqn (2.15), the eigenvalues and eigenfunctions of eqn (2.26) can be shown to be

$$\lambda_1=\epsilon_i^4+\frac{D_y}{D_x}\frac{a^4}{b^4}\epsilon_j^4 \tag{2.28}$$

and

$$\psi_1=\phi_i\phi_j \tag{2.29}$$

and that of eqn (2.27) to be

$$\lambda_2=\frac{Hb^2}{D_xa^2}m^2n^2\pi^4;\quad m=1,2,3,\ldots \quad n=1,2,3,\ldots \tag{2.30}$$

and

$$\psi_2=2\sin m\pi X\sin n\pi Y. \tag{2.31}$$

The D matrix is now constructed using the eqn (2.24) and the eigenfunctions given by eqns (2.29) and (2.30) as

$$D^{\alpha\alpha}=(\lambda^{\alpha}_{n_\alpha+1}-\lambda_\mu{}^{\alpha}),\ \alpha=1,2. \tag{2.32}$$

and

$$D^{12}=\{(\lambda^1_{n_1+1}-\lambda_\mu{}^1)^{1/2}(\psi_\mu{}^1,\ \psi_\mu{}^2)(\lambda^2_{n_2+1}-\lambda_\nu{}^2)^{1/2}\} \tag{2.33}$$

where

$$(\psi_\mu{}^1,\ \psi_\nu{}^2)=2\left\{\frac{m\pi}{\epsilon_\mu{}^2-m^2\pi^2}[1+(-1)^m(E_\mu\sinh\epsilon_\mu-\cosh\epsilon_\mu)]+\frac{m\pi}{\epsilon_\mu{}^2-m^2\pi^2}[1+(-1)^m(E_\mu\sin\epsilon_\mu-\cos\epsilon_\mu)]\right\}.$$
$$\left\{\frac{n\pi}{\epsilon_\nu{}^2-n^2\pi^2}[1+(-1)^n(E_\nu\sinh\epsilon_\nu-\cosh\epsilon_\nu)]+\frac{n\pi}{\epsilon_\nu{}^2-n^2\pi^2}[1+(-1)^n(E_\nu\sin\epsilon_\nu-\cos\epsilon_\nu)]\right\};\quad m=1,2,3,\ldots\quad n=1,2,3,\ldots \tag{2.34}$$

μ and ν correspond to the position in the D^{12} and D^{21} matrix.

3. DISCUSSION AND CONCLUSIONS

Figures 2–6 show that a linear correspondence exists between the fundamental plate frequency parameter and the ratio of the material properties. The parameter $\lambda_1=(\rho h\omega_1{}^2a^4)/D_x$ obtained by the Rayleigh–Ritz method increases with aspect ratio and its dependence on y-direction rigidity ratio D_y/H becomes more significant as shown in Figs. 5 and 6. In all of the above cases, the fundamental plate eigenvalue was associated with the first mode in both the x and y direction.

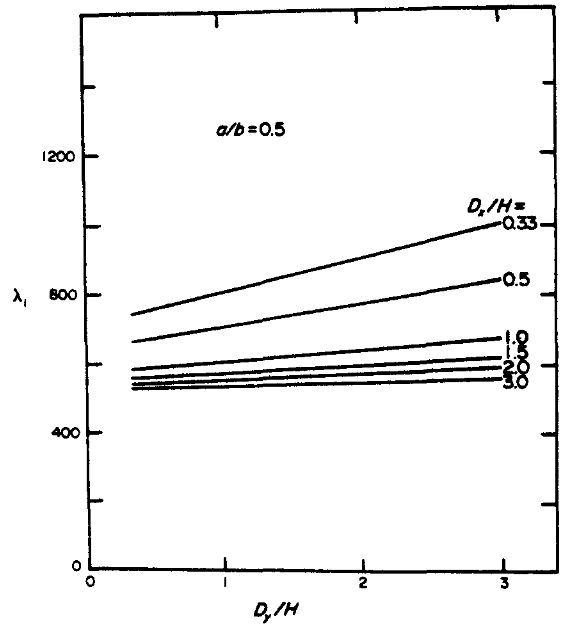

Fig. 2. Effect of rigidity variations on the fundamental frequency of a rectangular orthotropic clamped plate with an aspect ratio of 0.5.

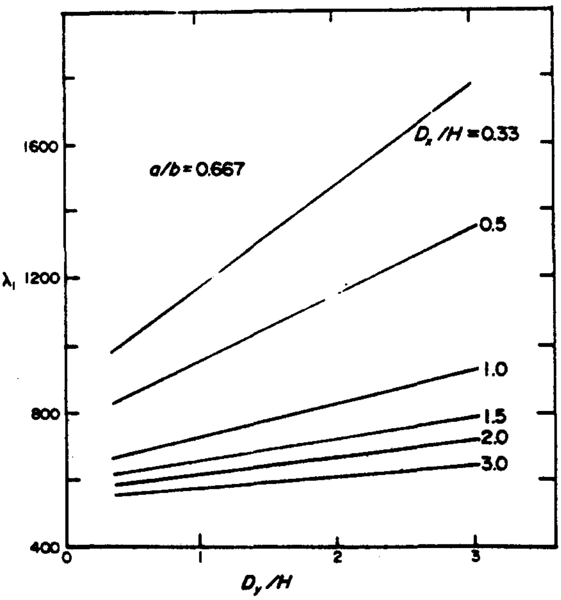

Fig. 3. Effect of rigidity variations on the fundamental frequency of a rectangular orthotropic clamped plate with an aspect ratio of 0.667.

Similar plots in Figs. 7–11 show the linear correspondence of the second eigenvalue for various rigidity ratios. However, a consistent straight line relationship is not maintained for all D_x/H ratios, as shown in Figs. 8–10. Beyond a specific point, the actual plate frequency based on the first mode in the x-direction and the second in the y-direction no longer yields the second lowest frequency. Hence, a second straight line of a decreased slope represents the second lowest plate frequency. The plot now represents the second mode in the x-direction and first in the y-direction. These plots are sometimes useful in critical design when it is important to know the frequency at which the mode shape flips from one to the other. The range of data plotted in the above graphs correspond to almost all the available orthotropic materials. Also, the linear plots in all of the above figures makes it possible to extrapolate the results.

In Table 1 the frequency parameter λ_1, of orthotropic, clamped square plates has been

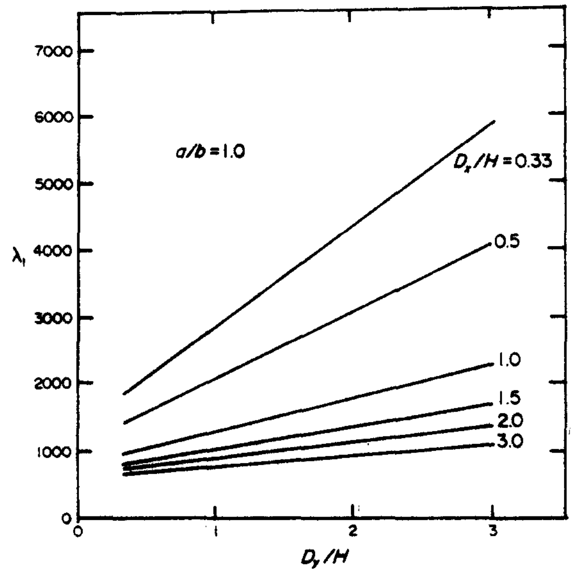

Fig. 4. Effect of rigidity variations on the fundamental frequency of a square orthotropic clamped plate.

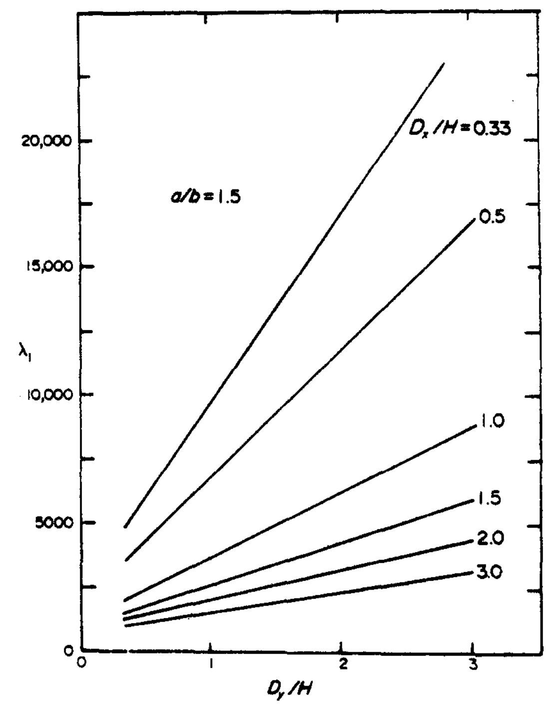

Fig. 5. Effect of rigidity variations on the fundamental frequency of a rectangular orthotropic clamped plate with an aspect ratio of 1.5.

Table 1. Comparison of fundamental frequency parameter for orthotropic square clamped plates

Rigidity Ratios		Frequency Parameter (λ_1)				
D_x/H	D_y/H	Jagadish	Claasen	Dickinson	Bazley & Fox	Present Work*
.5	.5	1 592.8	-	1 576.1	-	1 576.7
.5	1.0	2 093.3	-	2 082.8	-	2 083.4
.5	2.0	3 095.1	-	3 087.9	-	3 088.4
1.0	1.0	1 299.6	1 294.9	1 294.9	1 294.9	1 295.1
1.0	2.0	1 800.5	-	1 797.5	-	1 797.7

*Results based on Rayleigh-Ritz Method.

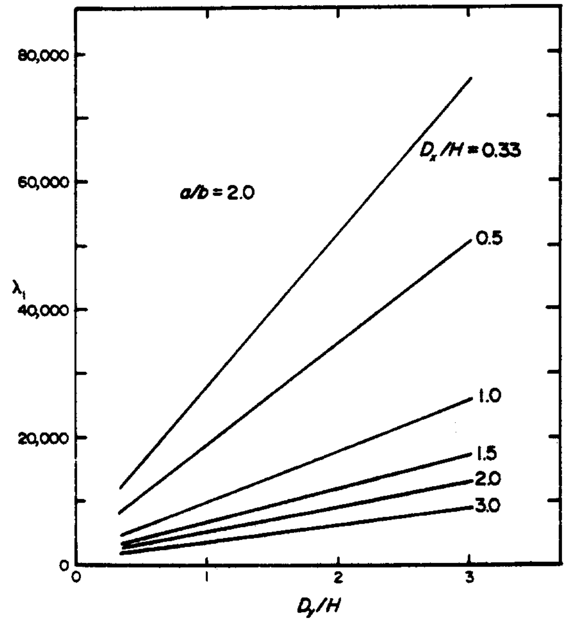

Fig. 6. Effect of rigidity variations on the fundamental frequency of a rectangular orthotropic clamped plate with an aspect ratio of 2.0.

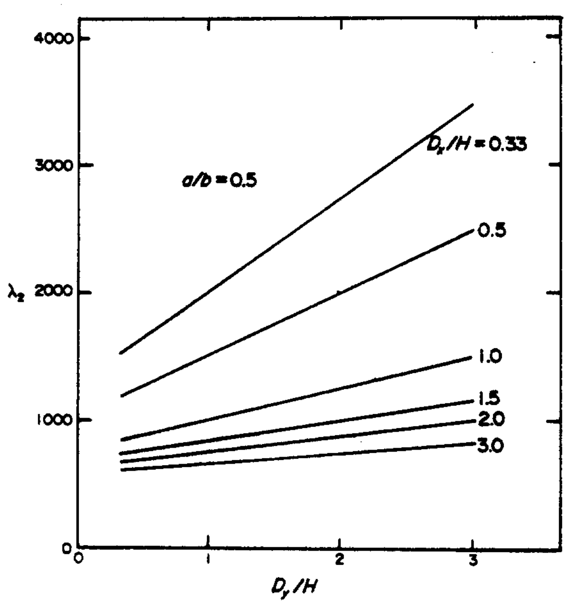

Fig. 7. Effect of rigidity variations on the second natural frequency of a rectangular orthotropic clamped plate with an aspect ratio of 0.5.

compared with special cases presented in the literature. Although Dickinson did not use an upper bound technique (his value increases with increase in function size) the values calculated in this study are within 0.04% of the reported results. Also all the results appear to be more accurate than those of Jagadish and Iyengar. Table 2 shows the comparison between the calculated frequency parameters and previously reported values for the first five modes of a special orthotropic square plate. In general, the calculated values for a plywood plate appear to agree well with Dickinson's and appear to be more accurate than Hearmon's.

Table 3 compares the upper and lower bounds of isotropic square clamped plates. The upper bound results of the present work appear to be better than those of Young and comparable to those of Bazely *et al.* Since no symmetry conditions were imposed, the second and third mode frequencies have also been reported in this study. The lower bounds are fairly close to the upper bounds indicating that the decompositional technique can be utilized with a complete

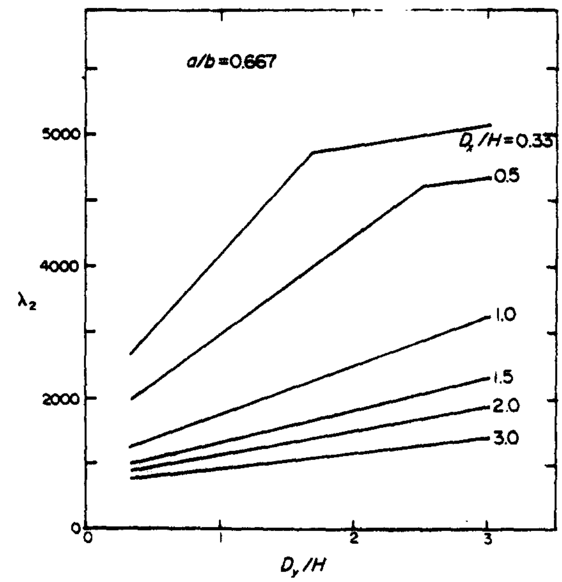

Fig. 8. Effect of rigidity variations on the second natural frequency of a rectangular orthotropic clamped plate with an aspect ratio of 0.667.

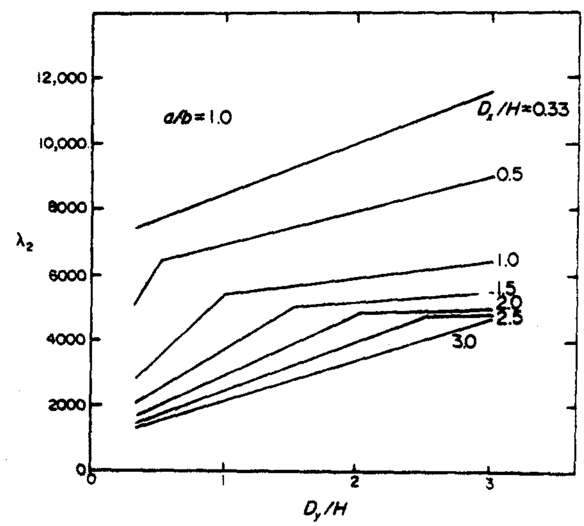

Fig. 9. Effect of rigidity variations on the second natural frequency of a square orthotropic clamped plate.

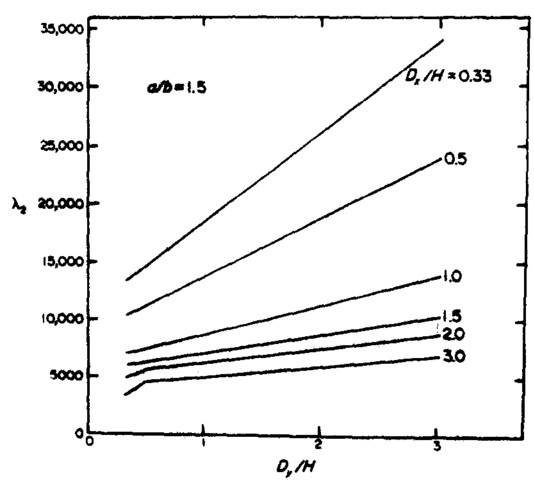

Fig. 10. Effect of rigidity variations on the second natural frequency of a rectangular orthotropic clamped plate with an aspect ratio of 1.5.

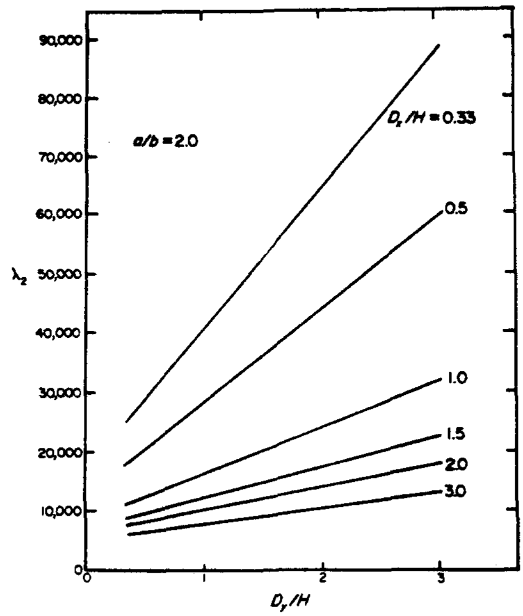

Fig. 11. Effect of rigidity variations on the second natural frequency of a rectangular orthotropic clamped plate with an aspect ratio of 2.0.

Table 2. Frequency of free vibration for orthotropic square clamped plate ($D_x/H = 1.543$, $D_y/H = 4.810$)

Mode No.	Frequency Parameter(λ)		
	Hearmon	Dickinson	Present Work*
1	2 272.7	2 254.5	2 254.5
2	6 186.7	6 087.4	6 086.3
3	13 825.	13 070.	13 069.
4	-	17 720.	17 718.
5	18 244.	18 366.	18 351.

*Results based on Rayleigh-Ritz Method.

Table 3. Comparison of the frequency parameters of the first four modes of an isotropic square clamped plate

Aspect Ratio A/B	Mode	Upper Bounds			Lower Bounds	
		Young	Bazley & Fox	Present Work*	Bazley & Fox	Present Work
1.0	1	1 295.2	1 294.9	1 295.1	1 294.3	1 286.7
	2	5 389.0	-	5 385.3	-	5 185.0
	3	11 722.	-	11 674.	-	-
	4	17 329.	17 313.	17 311.	17 293.	-
667	1	-	729.3	729.7	729.0	727.9
	2	-	-	1 739.8	-	1 655.3
	3	-	-	4 373.2	-	-
	4	-	4 425.3	4 430.3	4 417.1	-
50	1	-	604.1	604.2	603.9	599.1
	2	-	-	1 011.8	-	956.4
	3	-	2 004.5	2 022.8	2 000.3	-
	4	-	-	3 991.6	-	-

*Results based on Rayleigh-Ritz Method.

Table 4. Bounds for eigenvalues of orthotropic rectangular clamped plates

D_x/H	D_y/H	A/B	Order	Bound Estimates Upper	Bound Estimates Lower
1.543	4.810	0.5	1	646.4091	646.1528
			2	1422.3830	1391.4939
		1.0	1	2254.5420	2253.6489
			2	6086.3035	-
		1.5	1	8832.4839	8828.5156
			2	13329.83	-
		2.0	1	26227.98	26195.055
			2	31651.67	-
4.310	0.305	0.5	1	518.9465	516.3428
			2	579.9746	565.2004
		1.0	1	603.4624	599.6765
			2	1024.4738	974.0867
		1.5	1	833.5441	828.4370
			2	2434.6102	2327.4282
		2.0	1	1341.0440	1320.3628
			2	5379.3524	-
1.0	2.0	0.5	1	636.6271	634.9131
			2	1251.9036	1201.0317
		1.0	1	1797.6569	1787.2656
			2	5903.1162	-
		1.5	1	6228.9706	6208.5273
			2	11354.91	-
		2.0	1	17677.94	17593.883
			2	24223.39	-
1.0	3.0	0.5	1	668.4210	668.2830
			2	1490.6255	1442.0378
		1.0	1	2298.9819	2296.4839
			2	6410.9795	-
		1.5	1	8763.9808	8742.6250
			2	13898.00	-
		2.0	1	25687.92	25602.906
				32242.07	-
2.0	1.0	0.5	1	552.4357	549.8081
			2	756.9810	728.4688
		1.0	1	898.8284	893.6333
			2	2951.5581	2842.7612
		1.5	1	2100.3580	2091.9373
			2	6308.0068	-
		2.0	1	5093.0168	5079.2969
			2	10015.23	-
2.0	3.0	0.5	1	584.5791	584.4226
			2	996.4461	971.3010
		1.0	1	1400.7365	1398.5220
			2	5109.1492	-
		1.5	1	4636.1401	4635.3945
			2	8858.4351	-
		2.0	1	13103.92	13088.316
			2	18042.77	-

beam eigenfunction for the orthotropic plates. Although utilization of only six terms of a complete set of characteristic beam functions yielded accurate results for the upper bounds, corresponding accuracy was not achieved for the lower bounds. Difficulties attributed to squaring tangent terms in the actual computation of the lower bound values necessitated the limiting of the beam eigenfunctions to two. The instabilities are believed to be associated with evaluating the trigonometric functions, especially the tangent near $\pi/2$.

The bounds for clamped orthotropic plates for various rigidity and aspect ratios have been tabulated in Table 4. It is seen that the range for the "exact" eigenvalue varies from 0.02 to 3.6%.

Thus it appears that a combination of the Rayleigh–Ritz method for upper bound determination and the decomposition technique for lower bound estimation is a valuable procedure for the bracketing of the true natural frequencies of a wide class of vibration problems where an explicit closed form solution is impossible.

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